Midterm 2 Review

B - coords.

B - matrix

3.4: 4, 6, 8, 10, 12, 14, 20, 22, 24, 26, 28, 56, 62

5.1: 10, 15, 16, 26, 28

Gram-Schmidt

QR

5.2: 4, 6, 8, 18, 20, 22, $\underbrace{32, 34}_{\text{G-S in context}}$

5.3: 8, 10, 16, 20, 22, 26, 40

Least-squares

5.4: 20, 22, 24

T/F: Chapter 3: 1, 2, 3, 5, 6, 7, 10, 12, 16, 17, 18

Chapter 5: 2, 3, 4, 5, 6, 7, 10, 13, 14, 16, 20, 36, 39

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

To-Do: Chapter 5 computations

- Find ONB for $\text{im}(A)$ ✓
- Find QR decomp. for A ✓
- Find a matrix that repr. $\text{proj}_{\text{im}(A)}$ ✓
- Solve the least-squares problem ✓

$$A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

① Find $\underbrace{\text{ONB}}$ for $\mathrm{im}(A)$

orthonormal basis

↳ Gram-Schmidt

basis for subspace ⇝ ONB for that subspace

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} -(\mathrm{II}) \to \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix} -(\mathrm{I})$$

$$\to \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} /-2 \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\mathrm{im}(A) = \mathrm{span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}}_{\text{basis for } \mathrm{im}(A)}$$

Orthonormal basis:

↳ All the basis vectors are normalized, $\|\vec{v}\| = 1$

↳ All the basis vectors are mutually orthogonal, $\vec{v} \cdot \vec{w} = 0$

$$\begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \end{array}$$

$$\underbrace{\{\vec{v}_1, \dots, \vec{v}_n\}}_{\text{basis}} \rightsquigarrow \underbrace{\{\vec{w}_1, \dots, \vec{w}_n\}}_{\text{orthogonal}} \rightsquigarrow \underbrace{\{\vec{u}_1, \dots, \vec{u}_n\}}_{\text{orthonormal}}$$

$$\vec{u}_i = \frac{\vec{w}_i}{\|\vec{w}_i\|}$$

$$\vec{w}_1 = \vec{v}_1, \quad \vec{u}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{1+1+1}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\vec{v}_2 \cdot \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1} \vec{v}_2$$

$$= \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \left(-\frac{1}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -2/3 \\ -2/3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|} \qquad = \frac{2}{3}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\|w_2\| = \frac{2}{3}\sqrt{4+1+1} = \frac{2}{3} \cdot \sqrt{6} = \boxed{\frac{2\sqrt{6}}{3}}$$

$$\vec{u}_2 = \frac{\vec{w}_2}{\frac{2\sqrt{6}}{3}} = \frac{\cancel{3}}{\cancel{2}\sqrt{6}} \cdot \frac{\cancel{2}}{\cancel{3}}\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}$$

ONB for im(A)

QR decomp:

A, $n \times m$, w/ lin. ind. cols. then we can write

$$\underset{n\times m}{A} = \underset{n\times m}{Q}\ \underset{m\times m}{R}$$

Q: $Q^T Q = I$, whose cols. are an ONB for $\operatorname{im}(A)$

R: upper triangular $\begin{bmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{bmatrix}$ $R = Q^T A$

r_{ij} = the entry of R in the i^{th} row & j^{th} col.

$r_{ii} = \|\vec{w}_i\|$

$r_{ij} = \boxed{u_i \cdot v_j}$

(for $j > i$)

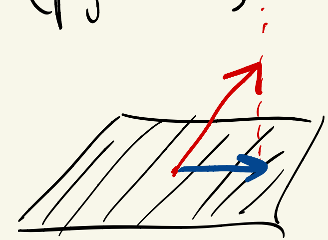
$$Q = \begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{3} & -1/\sqrt{3} \\ 0 & \frac{2\sqrt{6}}{3} \end{bmatrix}$$

$$R = \begin{bmatrix} \|w_1\| & u_1 \cdot v_2 \\ 0 & \|w_2\| \end{bmatrix}$$

③ Find the matrix that rep. $proj_{im(A)}$, P (projection)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad im(A) \text{ is a subspace of } \mathbb{R}^3$$

$\vec{v} \in \mathbb{R}^3$, $proj_{im(A)} \vec{v}$



(*) For a matrix A, if Q is the matrix whose cols. give an ONB for the $im(A)$, then $\boxed{P = QQ^T}$

$$P\vec{v} = proj_{im(A)} \vec{v} = proj_{\vec{u}_1} \vec{v} + proj_{\vec{u}_2} \vec{v}$$

$$= (\vec{u}_1 \cdot \vec{v})\vec{u}_1 + (\vec{u}_2 \cdot \vec{v})\vec{u}_2$$

$\boxed{P \text{ is symmetric}}$

$$
\begin{bmatrix} 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}
\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}
=
\begin{bmatrix} \frac{1}{3} + \frac{4}{6} & \overset{0}{\cancel{\frac{1}{3} - \frac{2}{6}}} & 0 \\ 0 & \overset{1/2}{\cancel{\frac{1}{3} + \frac{1}{6}}} & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}
$$

3×2 $\qquad$ 2×3 $\qquad$ 3×3

$$
= \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}}
$$

Want to solve: $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{array}\right]$$

$x_1 - x_2 = 0$

$x_1 - x_2 = 1$

no solution

Find the least-squares sol. to $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$A^TA\,\vec{x}^* = A^T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T A = \underset{2\times 3}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}} \overset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 2 \\ -1 & 3 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 3 & -1 & 2 \end{array}\right] \begin{array}{l} \\ +3(I) \end{array} \rightarrow \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 8 & 2 \end{array}\right] \begin{array}{l} /-1 \\ /8 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 1 & 1/4 \end{array}\right] \xrightarrow{+3(\mathrm{II})} \left[\begin{array}{cc|c} 1 & 0 & 3/4 \\ 0 & 1 & 1/4 \end{array}\right]$$

$$\boxed{\vec{x}^* = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

error : $\boxed{\| \vec{b} - A\vec{x}^* \|}$

$$\left\| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ -1/2 \\ 1/2 \end{bmatrix} \right\| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$$

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$T(\vec{x}) = A\vec{x}$

$B = \{ \vec{b}_1, \dots, \vec{b}_n \}$ basis of $\mathbb{R}^n$

$$[T]_B = \left[\boxed{[T(\vec{b}_1)]_B} \; [T(\vec{b}_2)]_B \; \cdots \; [T(\vec{b}_n)]_B \right]$$

① Compute $A\vec{b}_1$, this is a vector $\vec{v}_1$

② Write $\vec{v}_1$ in B-basis $\rightarrow$ $\underbrace{\left[\vec{b}_1 \cdots \vec{b}_n \,\middle|\, \vec{v}_1 \right]}$

Sol. to this augmented system is

T/F answers:

Chapter 3:

1) T 2) F 3) F 5) T 6) F 7) T 10) F 12) F

16) T 17) T 18) T

Chapter 5

2) T 3) T 4) F 5) T 6) T 7) F 10) T 13) T 14) F

16) F 20) T (QR) 36) T 39) T